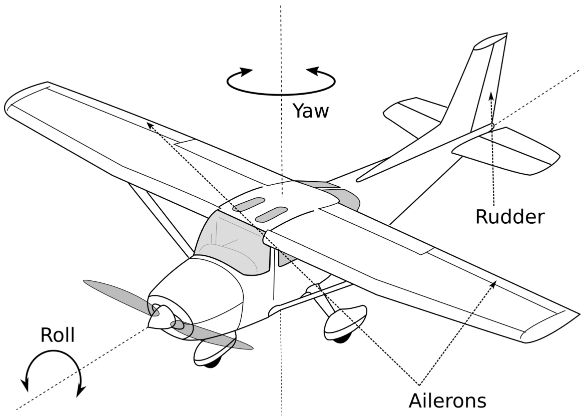
CALCULUS III: PROJECT 1

For the first project, we will investigate how an airplane propeller generates thrust and how that contributes to a particular turning tendency of single engine airplanes. A turning tendency of an airplane is an unintended yawing (rotating about a vertical axis) or rolling(rotating about the main horizontal axis) which the pilot needs to compensate for with the rudder or the ailerons respectively.



There are 4 turning tendencies which a single engine airplane experiences:

- *P-factor*[1] causes the airplane to yaw when the airplane is climbing due to the difference of the attack angle of the blades of the propeller.
- *Gyroscopic precession* causes the airplane to yaw to the left when the pilot raises the tail of a taildragger airplane during takeoff. This is the same effect that causes spinning tops to stay upright.
- *The torque effect* causes the airplane to roll to the left when the engine speed increases.
- *The spiraling slipstream effect* causes the airplane to yaw to the left due to the spiraling stream of air that hits the left side of the tail.

For this project we will investigate the p-factor.

[1] p-factor stands for propeller factor

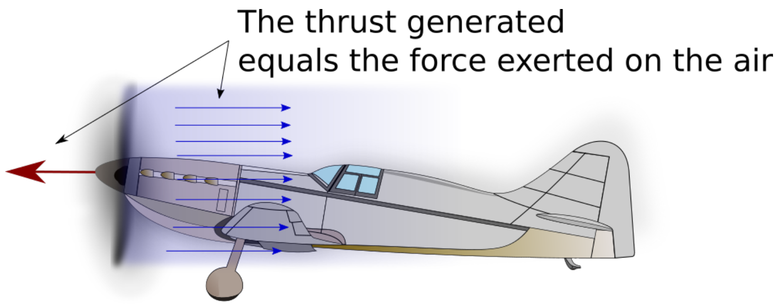
Instructions. This project consists of three required parts and one optional part. The optional part will not be graded. You may consult outside sources, especially in relation to the concepts from physics you will encounter while working on the project. You will work in groups of 3-4 students from your section set up by your TA. Only one person from each group needs to submit the project on Gradescope by the due date.

The parts that need to be completed are marked "Computation" and "The Question". Include as much explanations and computations in your answers as possible. Submitting only the answers will not receive full credit.

Besides submitting the projects on gradescope, you will be meeting with your TA for project check-ins. During these check-ins you will present your current progress on the project and get feedback from the TA. On the schedule tab of the course website, you will find when you should be presenting different parts of the project. Every student needs to present at least some part of the project.

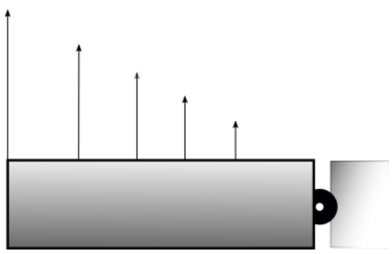
Part I. We will first investigate how the propeller produces the force accelerating the airplane forward. We will make many simplifications along the way in order to make the problem more manageable.

As our first simplification, we will assume that the thrust generated by the propeller is described entirely by the Newton's third law of motion, which states that for every force exerted by an object A on an object B, there is an equal in magnitude and opposite in direction force exerted by the object B on the object A.[2] In the case of the propeller, it implies that the force felt by the propeller is determined by the force the propeller exerts on the air in order to propel it backwards.

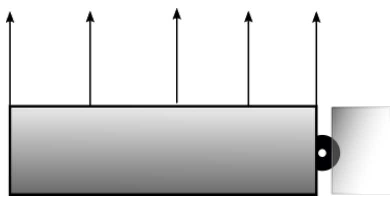


In order to compute how much air the propeller pushes backwards, let us consider each propeller blade separately. We will assume for simplicity that the blades are rectangular of length L and width W. We are interested in the instantaneous force exerted at a particular moment, so let us assume that at this moment the propeller blade is in the horizontal position to the left of the airplane as viewed by the pilot. We will also assume that the propeller is spinning clockwise as viewed by the pilot, which is the case for most airplanes.

As the blade rotates, the parts of the blade further away from the shaft are moving faster than the parts near the shaft. To simplify the problem, we will approximate the force resulting from a rotating blade by assuming that the blade moves uniformly upwards with the average speed of the blade.



The arrows indicate the instantaneous velocity of the blade as it rotates.

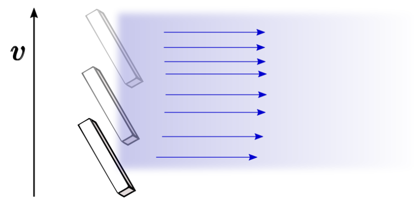


We will approximate the effect of a rotating blade by a blade moving upwards uniformly.

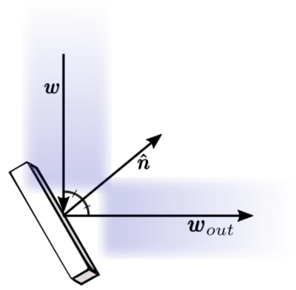
The diagrams are from the point of view of the pilot. Most single-engine airplanes' propellers rotate clockwise when viewed from the side of the pilot.

As the blade moves horizontally up with velocity $\boldsymbol{v}$, it displaces some amount of air towards the fuselage of the airplane as in the diagram below. It will turn out to be easier to consider the situation from the reference frame of the propeller blade, in which the

[2] A more precise description of the way propellers generate thrust would have to include Bernoulli's principle.



The air gets pushed to the right as the blade moves upwards



The air bounces off a stationary blade to the right.

The view is from the left of the airplane

wind blows downwards onto a stationary blade with velocity $\boldsymbol{w} = -\boldsymbol{v}$ and bounces off at velocity $\boldsymbol{w_{out}}$. Your first task will be to compute $\boldsymbol{w_{out}}$ in terms of $\boldsymbol{w}$ and the unit normal vector $\hat{\boldsymbol{n}}$.

Computation 1. Assuming that the collision is elastic so that $\|\boldsymbol{w}\| = \|\boldsymbol{w_{out}}\|$ and the reflection angle equals the incidence angle as in the figure above, find the expression for $\boldsymbol{w_{out}}$ in terms of $\boldsymbol{w}$, $\hat{\boldsymbol{n}}$ and the projection operation.

In order to compute the force on the propeller we need to recall the definition of momentum and its relation to force. Recall that momentum of an object of mass m and velocity $\boldsymbol{v}$ is defined by

$$\boldsymbol{p} = m\boldsymbol{v}$$

and Newton's second law of motion states that force equals instantaneous change of momentum

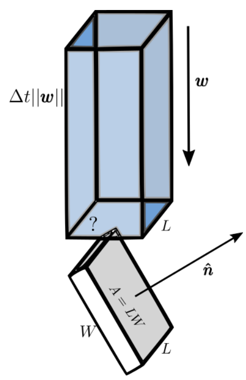
$$\boldsymbol{F} = m\frac{d}{dt}\boldsymbol{v} = \frac{d}{dt}\boldsymbol{p}.$$

We can thus compute the force exerted on the blade as the ratio of $\Delta\boldsymbol{p}_{blade}$, the change of momentum of the blade during some period of time Δt divided by Δt. By the law of conservation of momentum, or equivalently Newton's third law, $\Delta\boldsymbol{p}_{blade} = -\Delta\boldsymbol{p}_{air}$.

$$\boldsymbol{F}_{blade} = \frac{\Delta\boldsymbol{p}_{blade}}{\Delta t} = -\frac{\Delta\boldsymbol{p}_{air}}{\Delta t}$$

To find $\Delta\boldsymbol{p}_{air}$ we need to know the mass of air that gets reflected in time Δt, which is directly proportional to the volume of air reflected.

Computation 2. Determine the volume $V_{\Delta t}$ of air reflected in time Δt in terms of Δt, $\boldsymbol{w}, \hat{\boldsymbol{n}}$, and A, the area of the blade.



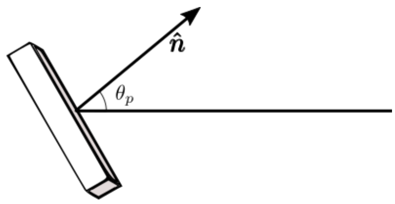
We are now in position to compute the force exerted on the blade in question. The momentum of the parcel of air that gets reflected in time Δt is $\rho_{air} V_{\Delta t} \boldsymbol{w}$ before reflection and $\rho_{air} V_{\Delta t} \boldsymbol{w}_{out}$ after, where ρ_{air} is the density of air. Therefore

$$\boldsymbol{F}_{blade} = -\rho_{air} \frac{V_{\Delta t}(\boldsymbol{w}_{out} - \boldsymbol{w})}{\Delta t}$$

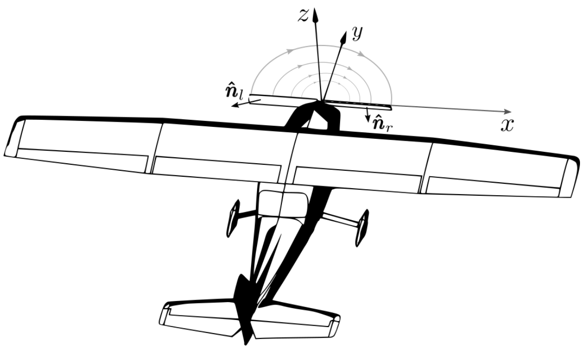
Computation 3. Find an expression for $\boldsymbol{F}_{blade}$ in terms of $\rho_{air}, \hat{\boldsymbol{n}}, A, \boldsymbol{w}$

We are now in the position to optimize our propeller blade.

Computation 4. Find the pitch θ_p, i.e. angle between propeller blades and a vertical plane, maximizing the component of $\boldsymbol{F}_{blade}$ in the forward direction. We will assume from now on that this is the pitch of our propeller.



Part II. In this part we will consider both blades of the propeller and compare the thrust produced by each one. Let us now introduce a coordinate system where z axis points up and the airplane is oriented toward the y axis. Denote by $\hat{\boldsymbol{n}}_l$ and $\hat{\boldsymbol{n}}_r$ the unit normal vectors to the left and right propeller blades respectively.



Computation 5. Find the vectors $\hat{\boldsymbol{n}}_l$ and $\hat{\boldsymbol{n}}_r$.

For the rest of this section, you will compute the forces and the thrust on the two propeller blades in three scenarios

(1) The airplane is standing still with the engine on, ready for takeoff.
(2) The airplane has accelerated on the runway and is about to lift off. It is still parallel to the ground.
(3) The airplane is still moving horizontally but has now raised its nose by θ_c in order to start climbing.

Analyze the three scenarios in exactly the same way we have done so far: assume the airplane and the propeller are stationary and it is the wind that exerts force on the propeller blades, which I will refer to by relative wind $\boldsymbol{w}$. In scenarios 2 and 3, there will be two contributions to the relative wind: the motion of the propeller blade and the motion of the plane. The relative wind will be the sum of the two contributions.

Let the speed of propeller blades be given by v_b in our approximation.[3] Let the speed of the airplane right before liftoff be v_p. You may assume for this section that v_b is much greater than v_p and therefore $\hat{\boldsymbol{n}} \cdot \boldsymbol{w} < 0$. The diagrams on the next page might be useful when thinking about scenarios 2 and 3.

Computation 6. Compute the force and thrust[4] on each blade of the propeller in the scenario (1) above. You will need to compute the velocity of the relative wind for each blade ($\boldsymbol{w}_l$ and $\boldsymbol{w}_r$). You can then use the formula from Computation 3 in order to find the force produced by each blade. Is the thrust produced by the two blades equal?

Computation 7. Compute the force and thrust on each blade of the propeller in the scenario (2) above. Repeat the steps of Computation 6 adjusting $\boldsymbol{w}_l$ and $\boldsymbol{w}_r$ accordingly. Is the thrust produced by the two blades equal?

Computation 8. Compute the force and thrust on each blade of the propeller in the scenario (3) above. Is the thrust produced by the two blades equal?

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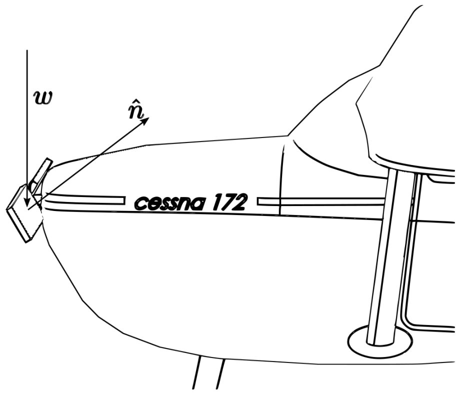
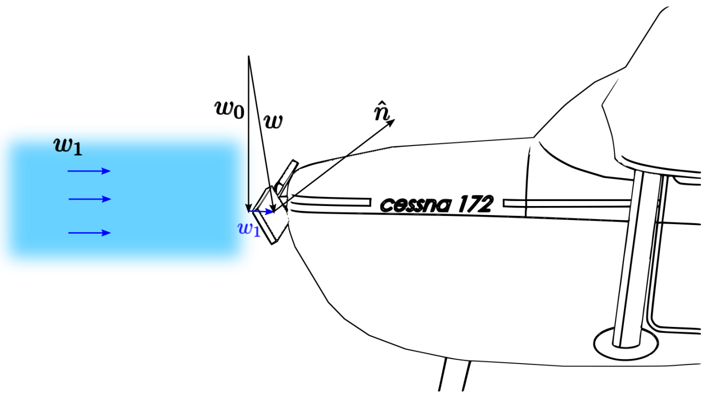
[3]We will compute v_b in terms of the length of the propeller blades and the rotation speed of the engine in Part III.

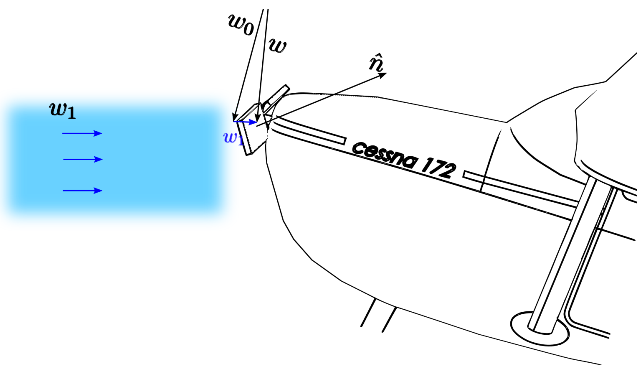
[4]The thrust is the component of force in the forward direction of the airplane



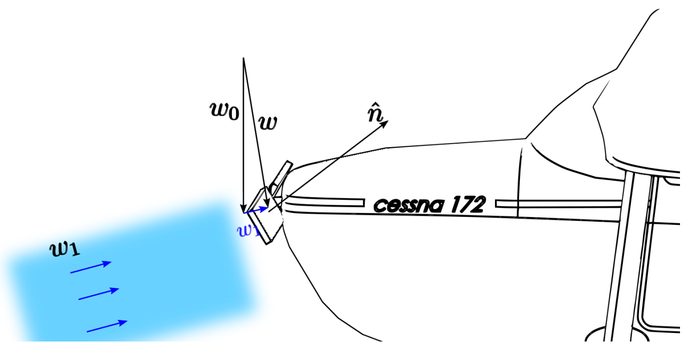
Computation 6: The relative wind $\boldsymbol{w}$ is due to the motion of the blades only



Computation 7: The relative wind $\boldsymbol{w}$ is due to the motion of the blade ($\boldsymbol{w}_0$) and the motion of the airplane ($\boldsymbol{w}_1$).



Same as in computation 7, but now the airplane has tilted



Instead of viewing the airplane as tilting, we can view the contribution of $\boldsymbol{w}_1$ as rotating.

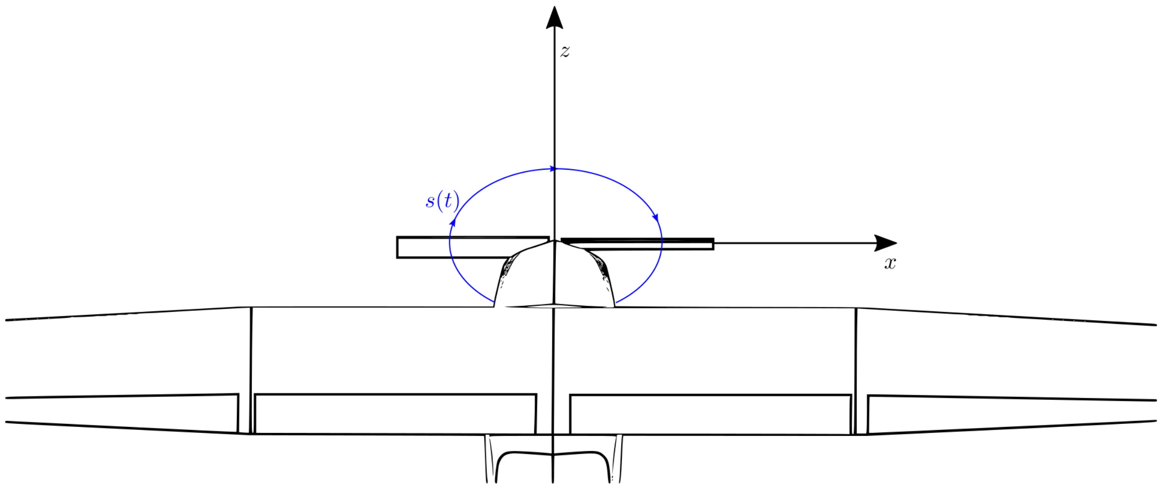
Computation 8

The Question: In which scenario does the p-factor contribute to the unintended yawing of the airplane and in which direction?

Part III.

Computation 9. For this computation, we will determine the average speed of the propeller blades in terms of the blade length L and the number of revolutions per minute R of the engine. As the propeller rotates, the middle of the blades follows a circular trajectory.

(1) Find the parametric equation $s(t)$, where t is measured in seconds, for the trajectory of the middle of the blade so that $s(0)$ is along the x axis.
(2) Find the time derivative $\frac{ds}{dt}(t)$ with respect to t.
(3) Evaluate the time derivative at $t = 0$ to find average velocity of the blade v_b.



Part IV. This part of the project is optional and will not be graded. Below you will find reasonable values for the constants you encountered in the first three sections of the project.

Computation 10. Using the constants below, compute the acceleration of the airplane when it just starts to accelerate and its acceleration right before liftoff. Using the average of the two accelerations, for how long does the airplane accelerate on the runway before lifting off?

$$\begin{aligned}
L &= 1m = \text{Length of the propeller blade} \\
W &= .1m = \text{Width of the propeller blade} \\
\rho_{air} &= 1.225kg/m^3 = \text{Density of air at sea level} \\
v_p &= 100mph = \text{The speed of the airplane right before takeoff} \\
R &= 2400RPM = \text{The rotation speed of the propeller} \\
M &= 1000kg = \text{The weight of the airplane}
\end{aligned}$$